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Gravitational radiation resistance, radiation damping and field fluctuations

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Abstract. Application is made of two different generalised fluctuation–dissipation theorems and their derivations to the calculation of the gravitational quadrupole radiation resistance using the radiation-reaction force given by Misner, Thorne and Wheeler and the usual tidal force on one hand and the tidal force and the free gravitational radiation field on the other hand. The quantum-mechanical version (including thermal generalisations) of the well known classical quadrupole radiation damping formula is obtained as a function of the radiation resistance.

1. Introduction

Following the derivation of the famous classical quadrupole formula for gravitational radiation with the help of a radiation-reaction force (obtained by matched asymptotic expansion and therefore often criticised†) by Misner *et al* (1973) and the corresponding quantum-mechanical formula derived by Schäfer and Dehnen (1980a, b) using the ratio of the Einstein coefficients for spontaneous emission and absorption together with a calculation of the absorption and a path integral approach, respectively, it is interesting to compare these approaches to the gravitational radiation damping problem more closely than only from their identity with respect to the correspondence principle applied to the radiation formulae as done by Schäfer and Dehnen (1980a, b). This is a much more interesting problem as the two damping mechanisms seem to be completely different: one uses the radiation-reaction force (caused by self-coupling), the other the influence of the vacuum fluctuations of the gravitational wave field.

To achieve our aim, we want in particular to make use of two beautiful papers, both of which give generalised fluctuation–dissipation theorems: one by Callen and Welton (1951) and the other by Feynman and Vernon (1963). The crucial points concerning these fluctuation–dissipation theorems are that, in deriving both, classical and quantum-mechanical equations are involved, the classical limit can be performed immediately, and they are dual to one another in some sense. As we shall see, in our case the difference between the fluctuation–dissipation theorems is a purely conceptual one. In the formalism of Callen and Welton (1951) the fluctuations of, say, the gravitational field potential in an isolated quantised matter system are related to the dissipative part of the impedance of the isolated system (at thermal equilibrium), which is determined by the radiation-reaction force and the linear gravitational wave coupling (tidal force),

† See § V of the recent review paper by Thorne (1980).

whereas in the formalism of Feynman and Vernon (1963) the fluctuations of the gravitational field potential in an unisolated quantum-mechanical system are related to the dissipative part of the impedance for the linear coupling (tidal force) of free gravitational waves (quantised and at thermal equilibrium) to the system. As they should, both formulae turn out to be identical† and show most clearly the essential identity of the two damping mechanisms mentioned above together with the correctness of the classical radiation-reaction force for calculating resistances.

In our choice for the model of the matter system we will restrict ourselves for simplicity to the classical and quantum-mechanical vibrators already treated in detail by Misner *et al* (1973) and Schäfer and Dehnen (1980a) respectively.

2. Application of the theory of Callen and Welton

First we give a brief outline of the main features of the theory of Callen and Welton.

We take as the Hamiltonian

$$H = H_0(x, p) - V(t)\boldsymbol{Q}(x, p) \tag{2.1}$$

where H_0 is the unperturbed Hamiltonian of the matter system which is assumed to be bounded from below with *densely* distributed eigenvalues (dissipative system), $\boldsymbol{Q}$ is a function of the coordinates and momenta of the matter system and V is a time-dependent (we choose a periodic time dependence with period ω) potential or force function which measures the instantaneous magnitude of the perturbation. Then, if the quantum-mechanical system is left in thermal equilibrium with no applied force, for the fluctuations of the hypothetical operator $V_\omega = Z(\omega)\dot{\boldsymbol{Q}}_\omega$ (its definition is given in terms of Fourier transforms) the following holds (cf also Bernard and Callen 1959):

$$\langle V(t)V(t+\tau)\rangle = \frac{2}{\pi}\int_0^\infty R(\omega)E(\omega, \beta)\cos\omega\tau\, \mathrm{d}\omega \tag{2.2}$$

where

$$E(\omega, \beta) = \tfrac{1}{2}\hbar\omega \coth \tfrac{1}{2}\hbar\beta\omega = \tfrac{1}{2}\hbar\omega + \hbar\omega/(\exp(\hbar\beta\omega) - 1)$$

with temperature β^{-1} and $R(\omega)$ is a resistance, the real or dissipative part of the impedance $Z(\omega)$. $Z(\omega)$ is defined by the classical relation ($\boldsymbol{Q}$ is to be taken here as a c-number and written, together with V from (2.1), in standard complex notation)

$$V = Z(\omega)\dot{\boldsymbol{Q}}, \tag{2.3}$$

valid for linear systems (notice the linearity of V of our perturbation term in equation (2.1)).

The classical limit ($\beta \to 0$) of equation (2.2) simply reads

$$\langle V(t)V(t+\tau)\rangle = \frac{2}{\pi}\beta^{-1}\int_0^\infty R(\omega)\cos\omega\tau\, \mathrm{d}\omega. \tag{2.4}$$

Considering equation (2.2), it is worthwhile pointing out a very interesting property: although the function V in (2.1) is a purely classical one, its fluctuations, because of $E(\omega, \beta)$, are uniquely quantum mechanical in origin and evidently arise from the quantised matter system. Next we turn to the application.

† The existence of temperature and the internal consistency of quantum mechanics demand this as a sort of compatibility condition.

As shown in the paper by Schäfer and Dehnen (1980a), the Hamiltonian of a bound matter system interacting with gravitational waves on the quadrupole approximation level can be written as

$$H = H_0(x, p) + \tfrac{1}{6}c^2 R_{i0j0}(t) Q^{ij} \tag{2.5}$$

where H_0 is again the unperturbed Hamiltonian, R_{i0j0} are the non-zero components (modulo symmetry properties) of the trace-free curvature tensor for the gravitational wave field and Q^{ij} is the mass-quadrupole tensor of the matter system which is defined by

$$Q^{ij} = \int \rho(3x^i x^j - r^2 \delta^{ij})\, \mathrm{d}^3 x \tag{2.6}$$

(ρ is the mass density, $i, j = 1, 2, 3$, c is the velocity of light). The perturbation term in equation (2.5) results from the tidal forces.

For our matter model (vibrator) we choose the following data†: two equal point masses with reduced mass μ, rest distance of the two masses $2L$, eigenfrequency ω_0, fixed unit vector in the mass–mass direction n^i, relative coordinate (momentum) of the two masses $x(p)$. Then, applied to our vibrator, equation (2.5) reads

$$H = (p^2/2\mu) + \tfrac{1}{2}\mu\omega_0^2(x - 2L)^2 + c^2 R_{i0j0} n^i n^j \tfrac{1}{2}\mu x^2. \tag{2.7}$$

Comparing equation (2.7) with equation (2.1) we make the identifications $V = -c^2 R_{i0j0} n^i n^j$ and $\boldsymbol{Q} = \frac{1}{2}\mu x^2$‡. Then, with equation (2.2) we get (from now on we choose $\tau = 0$)

$$\langle (c^2 R_{i0j0} n^i n^j)^2 \rangle = \frac{2}{\pi} \int_0^\infty R(\omega) E(\omega, \beta)\, \mathrm{d}\omega. \tag{2.8}$$

For the determination of $R(\omega)$ we proceed as follows. The equation of motion corresponding to the Hamiltonian (2.7) has the form

$$\ddot{x} + \omega_0^2(x - 2L) = -c^2 R_{i0j0} n^i n^j x \tag{2.9}$$

(a dot means derivative with respect to the time t). We must add to the right-hand side of this equation the radiation-reaction acceleration given by Misner *et al* (1973)

$$-\frac{2}{15}\frac{G}{c^5}\left(\frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}\right) x n^i n^j = -\frac{4}{15}\frac{G}{c^5}\mu x \frac{\mathrm{d}^5 x^2}{\mathrm{d}t^5} \tag{2.10}$$

where the vibrator data are already inserted. This results in

$$\ddot{x} + \omega_0^2(x - 2L) + \frac{4}{15}\frac{G}{c^5}\mu x \frac{\mathrm{d}^5 x^2}{\mathrm{d}t^5} = -c^2 R_{i0j0} n^i n^j x. \tag{2.11}$$

After performing the transformation $x = 2L(1 + \xi)$ and paying attention to $\xi \ll 1$ we get from equation (2.11) the equation of motion

$$\ddot{\xi} + \omega_0^2 \xi + \frac{32}{15}\frac{G}{c^5}\mu L^2 \frac{\mathrm{d}^5 \xi}{\mathrm{d}t^5} = -c^2 R_{i0j0} n^i n^j. \tag{2.12}$$

† The centre-of-mass data can be neglected.

‡ The reader may wonder about the densely distributed energy eigenvalues of our vibrator. This will effectively be achieved by the damping term (2.10) (see, e.g., Feynman and Hibbs 1965, pages 149 ff and page 161).

Now we make the ansatz $\xi = \xi_0 \, e^{-i\omega t}$ (R_{i0j0} oscillates with frequency ω!) and obtain from equation (2.12)

$$-c^2 R_{i0j0} n^i n^j = \left((\omega_0^2 - \omega^2) - i \frac{32}{15} \frac{G}{c^5} \mu L^2 \omega^5 \right) \xi. \tag{2.13}$$

With $d(\frac{1}{2}\mu x^2)/dt = 4\mu L^2 \dot{\xi} = -i4\mu L^2 \omega \xi$ and with respect to equation (2.3) and the substitution for $\boldsymbol{Q}$ defined above, equation (2.13) gives for the resistance $R(\omega)$

$$R(\omega) = \frac{8}{15} \frac{G}{c^5} \omega^4. \tag{2.14}$$

If we insert expression (2.14) into equation (2.8) we obtain the result

$$\langle (c^2 R_{i0j0} n^i n^j)^2 \rangle = \frac{16}{15\pi} \frac{G}{c^5} \int_0^\infty \omega^4 E(\omega, \beta) \, d\omega \tag{2.15}$$

or, after taking the average (see, e.g., equations (4.7) and (4.11) of Schäfer and Dehnen (1980a)) over all directions and polarisation states of the wave field (represented here by R_{i0j0}),

$$\langle R_{i0j0} R^{i0j0} c^4 \rangle = \frac{8}{\pi} \frac{G}{c^5} \int_0^\infty \omega^4 E(\omega, \beta) \, d\omega. \tag{2.16}$$

In the transverse traceless gauge of the radiation field h_{ij} the relation between R_{i0j0} and h_{ij} reads $c^2 R_{i0j0} = -\frac{1}{2}\ddot{h}_{ij}$ and between the energy density u of the radiation field and h_{ij}, $u = (c^2/32\pi G)\langle \dot{h}_{ij} \dot{h}^{ij} \rangle$. Taking into account these relations, it follows from equation (2.16) that

$$\langle h_{ij} h^{ij} \rangle = \frac{32}{\pi} \frac{G}{c^5} \int_0^\infty E(\omega, \beta) \, d\omega \tag{2.17}$$

and

$$u = \frac{1}{\pi^2 c^3} \int_0^\infty \omega^2 E(\omega, \beta) \, d\omega. \tag{2.18}$$

With regard to the definition of $E(\omega, \beta)$, the expression (2.18) is the well known Planck formula together with the zero-point contribution. This was to be expected because photons and gravitons have the same number of polarisation states.

3. Application of the theory of Feynman and Vernon

As in § 2, we want first to give a brief outline of the theory as far as we are concerned with it.

Let the total Lagrangian L be of the form

$$L = L_0(x, \dot{x}) + L_r(q, \dot{q}) + qQ(x, \dot{x}) \tag{3.1}$$

where L_0 is the unperturbed Lagrangian for the matter system, L_r is the Lagrangian of the dissipative system for which we choose a free external radiation field (densely distributed energy eigenvalues!) and qQ describes the q-linear coupling between both systems whereby Q is a function of the coordinates and velocities of the matter system.

Then the probability that the matter system makes a transition from a state $\Psi_n(x_\tau)$ at $t=\tau=0$ to $\Psi_m(x_T)$ at $t=T$ can be written (cf also Feynman and Hibbs 1965, ch 12)

$$P_{nm}=\int \Psi_m^*(x_T)\Psi_m(x'_T)\exp\Big(\frac{\mathrm{i}}{\hbar}(S_0(x)-S_0(x'))\Big)F(Q,Q')$$

$$\times\Psi_n^*(x'_\tau)\Psi_n(x_\tau)\mathrm{D}x(t)\mathrm{D}x'(t)\,\mathrm{d}x_\tau\,\mathrm{d}x'_\tau\,\mathrm{d}x_T\,\mathrm{d}x'_T \tag{3.2}$$

where $S_0(x)=\int_0^T L_0(x,\dot{x})\,\mathrm{d}t$ is the action corresponding to L_0, $F(Q,Q')$ is the influence functional which incorporates all effects of the external influences on Q and $\int \mathrm{D}\ldots$ means functional integration. The influence functional has the form

$$F(Q,Q')=\exp(\mathrm{i}\phi(Q,Q')) \tag{3.3a}$$

with the influence phase $\phi(Q,Q')$ given by

$$\phi(Q,Q')=\frac{\mathrm{i}}{2\hbar}\int_0^T\int_0^t (Q_t-Q'_t)(Q_sF^*(t-s)-Q'_sF(t-s))\,\mathrm{d}s\,\mathrm{d}t. \tag{3.3b}$$

If the q-system modes are thermally distributed, then

$$F(t)=+\frac{2}{\pi}\int_0^\infty \mathrm{Re}(\omega Z_\omega)^{-1}[1+2(\exp(\beta\hbar\omega)-1)^{-1}]\cos\omega t\,\mathrm{d}\omega$$

$$+\frac{2}{\pi}\mathrm{i}\int_0^\infty \mathrm{Re}(\omega Z_\omega)^{-1}\sin\omega t\,\mathrm{d}\omega \tag{3.4}$$

where Z_ω is a classical impedance function which relates the reaction of q to the applied force Q according to†

$$Z_\omega=-Q_\omega/\mathrm{i}\omega q_\omega. \tag{3.5}$$

The definitions are

$$q_\omega=\int_0^\infty q(t)\,\mathrm{e}^{+\mathrm{i}\omega t}\,\mathrm{d}t \qquad \text{and} \qquad Q_\omega=\int_0^\infty Q(t)\,\mathrm{e}^{+\mathrm{i}\omega t}\,\mathrm{d}t.$$

For the determination of Z_ω the classical equation of motion which follows from the Lagrangian (3.1)

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}}-\frac{\partial}{\partial q}\right)L_r(q,\dot{q})=Q \tag{3.6}$$

has to be solved under the initial conditions at $t=\tau=0$: $q(0)=\dot{q}(0)=0$. If we take, for example,

$$L_r(q,\dot{q})=\tfrac{1}{2}m(\dot{q}^2-\omega_0^2q^2), \tag{3.6a}$$

then we obtain as the equation of motion

$$m(\ddot{q}+\omega_0^2q)=Q \tag{3.6b}$$

with the solution

$$q(t)=(m\omega_0)^{-1}\int_0^t Q(s)\sin\omega_0(t-s)\,\mathrm{d}s, \tag{3.6c}$$

† Because of reciprocity relations, the impedance Z_ω is linked, of course, with the impedance $Z(\omega)$ defined by (2.3).

or, in terms of Fourier transforms,

$$q_\omega = \frac{-Q_\omega}{m[(\omega + \mathrm{i}\epsilon)^2 - \omega_0^2]}. \tag{3.6d}$$

The correlation function for q, $\langle q(t)q(t+s)\rangle$, is related to $F(t)$ according to†

$$\langle q(t)q(t+s)\rangle = \tfrac{1}{2}\hbar \operatorname{Re} F(s). \tag{3.7}$$

In second-order approximation with respect to the potential q in the coupling term of equation (3.1), P_{nm} from equation (3.2) reads

$$P_{nn} = 1 - \sum_{k,\omega_{nk}>0} 2T(\hbar\omega_{nk})^{-1}|Q_{nk}|^2 \operatorname{Re}(Z_{\omega_{nk}})^{-1} \frac{\mathrm{e}^{\hbar\beta\omega_{nk}}}{\mathrm{e}^{\hbar\beta\omega_{nk}}-1}$$
$$- \sum_{k,\omega_{nk}<0} 2T(\hbar\omega_{kn})^{-1}|Q_{nk}|^2 \operatorname{Re}(Z_{\omega_{kn}})^{-1} \frac{1}{\mathrm{e}^{\hbar\beta\omega_{kn}}-1} \tag{3.8a}$$

(spontaneous and induced decay)

$$P_{nm} = 2T(\hbar\omega_{nm})^{-1}|Q_{nm}|^2 \operatorname{Re}(Z_{\omega_{nm}})^{-1} \frac{\mathrm{e}^{\hbar\beta\omega_{nm}}}{\mathrm{e}^{\hbar\beta\omega_{nm}}-1} \qquad \text{for } \omega_{nm}>0 \tag{3.8b}$$

(spontaneous and induced emission)

$$P_{mn} = 2T(\hbar\omega_{nm})^{-1}|Q_{nm}|^2 \operatorname{Re}(Z_{\omega_{nm}})^{-1} \frac{1}{\mathrm{e}^{\hbar\beta\omega_{nm}}-1} \qquad \text{for } \omega_{nm}>0 \tag{3.8c}$$

(absorption) where the definitions $\hbar\omega_{nk} = E_n - E_k$ (E_n is the energy eigenvalue of the state Ψ_n) and $Q_{nk} = \int \Psi_n^*(x) Q \Psi_k(x)\,\mathrm{d}x$ are used.

At this point we want to make a short digression to the thermal matter of § 2 to show a consistency with the thermal radiation calculations of this section. As can be inferred, e.g. from the book by Weinberg (1972), a relative probability for spontaneous and induced emission of radiation from thermal matter with temperature β'^{-1} can be written with, say, our P_{nm} as

$$p_{nm} = P_{nm}\,\mathrm{e}^{-\hbar\beta'\omega_{nm}} \qquad (\omega_{nm}>0) \tag{3.8b'}$$

and, correspondingly, for absorption

$$p_{mn} = P_{mn} \qquad (\omega_{mn}<0). \tag{3.8c'}$$

In the case of thermal equilibrium between matter and radiation we must have, of course, $p_{nm} = p_{mn}$. From this, using equations (3.8b), (3.8c), (3.8b') and (3.8c'), it follows that $\beta = \beta'$, thus proving the consistency.

Now we will apply the stated formalism. The Lagrangian corresponding to the Hamiltonian (2.5) has the form

$$L = L_0(x, \dot{x}) - \tfrac{1}{6}c^2 R_{i0j0}(t) Q^{ij}. \tag{3.9}$$

Using the gauge degree of freedom of adding a total time derivative to a Lagrangian, an equivalent Lagrangian to (3.9) is (cf Schäfer and Dehnen 1980a)

$$L = L_0(x, \dot{x}) + \tfrac{1}{12} h_{ij}(t) \ddot{Q}^{ij} \tag{3.10}$$

(h_{ij} is again given in the transverse traceless gauge). With the Lagrangian of the free

† To be precise, Feynman and Vernon use here the thermal part of F only.

gravitational radiation field (cf, e.g., Schäfer and Dehnen 1980b)

$$L_r(q, \dot{q}) = -\frac{c^4}{64\pi G}\int h_{ij|\mu} h^{ij|\mu}\, \mathrm{d}^3x \tag{3.11}$$

($\mu = 0, 1, 2, 3$; metric signature $+2$; $x^0 = ct$) our total Lagrangian, corresponding to equation (3.1), can be written as

$$L = L_0(x, \dot{x}) - \frac{c^4}{64\pi G}\int h_{ij|\mu} h^{ij|\mu}\, \mathrm{d}^3x + \tfrac{1}{12}h_{ij}(t)\ddot{Q}^{ij}. \tag{3.12}$$

The second time derivatives in equation (3.12) do not cause trouble with respect to equation (3.1) because in our approximation the time derivatives are generated by H_0 (corresponding to L_0) so that $\ddot{Q}^{ij}$ is effectively a function of the coordinates and velocities alone. Now we decompose the radiation field into normal modes $q^{(\alpha i)}_{(\boldsymbol{k})}$ with $q^{(\alpha i)}_{(-\boldsymbol{k})} = q^{(\alpha i)}_{(\boldsymbol{k})}(-1)^{i-1}$, $q^{(\alpha i)}_{(\boldsymbol{k})}$ real, and $i = 1, 2$; $\alpha 1 = 1$, $\alpha 2 = 3$ for $\alpha = 1$, and $\alpha 1 = 2$, $\alpha 2 = 4$ for $\alpha = 2$:

$$h_{ij}(t, \boldsymbol{x}) = \frac{(64\pi G)^{1/2}}{c}\sum_\alpha \int \frac{\mathrm{d}^3k}{16\pi^3}\, e^{(\alpha)}_{ij}\, (q^{(\alpha 1)}_{(\boldsymbol{k})} \cos(\boldsymbol{kx}) + q^{(\alpha 2)}_{(\boldsymbol{k})} \sin(\boldsymbol{kx})) \tag{3.13}$$

where $e^{(\alpha)}_{ij}$ are real polarisation tensors having the properties

$$e^{(\alpha)}_{ij}k^j = 0 \qquad e^{(\alpha)i}_{i} = 0 \qquad e^{(\alpha)ij}e^{(\beta)}_{ij} = \delta^{\alpha\beta} \qquad e^{(\alpha)}_{ij} = e^{(\alpha)}_{ji}.$$

Inserting (3.13) into (3.12) results in

$$\begin{aligned} L = L_0(x, \dot{x}) &+ \tfrac{1}{2}\sum_{\alpha i}\int \frac{\mathrm{d}^3k}{16\pi^3}[(\dot{q}^{(\alpha i)}_{(\boldsymbol{k})})^2 - k^2c^2(q^{(\alpha i)}_{(\boldsymbol{k})})^2] \\ &+ \frac{(64\pi G)^{1/2}}{12c}\sum_\alpha \int \frac{\mathrm{d}^3k}{16\pi^3}\, e^{(\alpha)}_{ij}\ddot{Q}^{ij}(q^{(\alpha 1)}_{(\boldsymbol{k})} \cos(\boldsymbol{kx}_0) + q^{(\alpha 2)}_{(\boldsymbol{k})} \sin(\boldsymbol{kx}_0)). \end{aligned} \tag{3.14}$$

In the last term we have made use of the quadrupole approximation, $\boldsymbol{kx} = \boldsymbol{kx}_0$ ($\boldsymbol{x}_0$ is the position of the centre of mass of the matter system).

Applied to our vibrator, we get

$$e^{(\alpha)}_{ij}\ddot{Q}^{ij} = e^{(\alpha)}_{ij}n^in^j\frac{\mathrm{d}^2}{\mathrm{d}t^2}(3\mu x^2)$$

and together with the Lagrangian (3.14) after conveniently changing the variables $q^{(\alpha i)}_{(\boldsymbol{k})}$ according to

$$q'^{(\alpha l)}_{(\boldsymbol{k})} = q^{(\alpha l)}_{(\boldsymbol{k})}\frac{(64\pi G)^{1/2}}{12c}\, e^{(\alpha)}_{ij}n^in^jA_l(\boldsymbol{k})$$

with

$$A_l(\boldsymbol{k}) = \delta_{l1}\cos(\boldsymbol{kx}_0) + \delta_{l2}\sin(\boldsymbol{kx}_0),$$

the equations of motion corresponding to equations (3.6) and (3.6*b*) are

$$\frac{144c^2}{64\pi G}(e^{(\alpha)}_{ij}n^in^jA_l(\boldsymbol{k}))^{-2}(\ddot{q}'^{(\alpha l)}_{(\boldsymbol{k})} + k^2c^2q'^{(\alpha l)}_{(\boldsymbol{k})}) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(3\mu x^2). \tag{3.15}$$

Comparing equation (3.15) with equation (3.6*b*) and using the solution (3.6*d*) we obtain as the solution for the impedance defined in equation (3.5)

$$(Z_\omega^{(\alpha l)}(\boldsymbol{k}))^{-1}=\frac{64\pi G}{144c^2}(e_{ij}^{(\alpha)}n^in^jA_l(\boldsymbol{k}))^2\frac{\mathrm{i}\omega}{(\omega+\mathrm{i}\epsilon)^2-k^2c^2}. \tag{3.16}$$

It is almost superfluous to say that $[(\omega+\mathrm{i}\epsilon)^2-k^2c^2]^{-1}$ is the Fourier transform of a retarded Green function, which, of course, is a typical classical function in contrast, for example, to the Feynman propagator which is fundamental in quantum mechanics. Its Fourier transform would look like $(\omega^2-k^2c^2+\mathrm{i}\epsilon)^{-1}$.

For the total effect of all possible gravitational waves† we get

$$\begin{aligned}(Z_\omega)^{-1}&=\sum_{\alpha i}\int\frac{\mathrm{d}^3k}{16\pi^3}(Z_\omega^{(\alpha i)}(\boldsymbol{k}))^{-1}\\ &=\frac{1}{45}\frac{G}{c^2}\left[\left(\frac{2}{3}\frac{\omega^2}{c^3}\right)+\mathrm{i}\left(\frac{4\omega}{3\pi c^3}\right)\int_0^\infty \mathrm{d}(kc)\frac{(kc)^2}{\omega^2-(kc)^2}\right]\end{aligned} \tag{3.17}$$

where use has been made of the identity

$$\lim_{\epsilon\to 0}\frac{1}{(\omega+\mathrm{i}\epsilon)^2-k^2c^2}=\frac{1}{\omega^2-k^2c^2}-\frac{\mathrm{i}\pi}{2kc}(\delta(\omega-kc)-\delta(\omega+kc)),$$

the definition of $A_l(\boldsymbol{k})$ and the relation

$$\sum_\alpha\int(e_{ij}^{(\alpha)}n^in^j)^2\,\mathrm{d}\Omega=\tfrac{8}{5}\pi\times\tfrac{2}{3}$$

whereby $\mathrm{d}\Omega=\mathrm{d}^3k/k^2\,\mathrm{d}k$.

By comparing the interaction terms in equations (3.1) and (3.14) and considering equation (3.13) and the definition of $q'^{(\alpha i)}_{(k)}$, we obtain with the help of equation (3.4) and the identification $Q\equiv \mathrm{d}^2(3\mu x^2)/\mathrm{d}t^2$ from equation (3.7) the relation

$$\frac{\langle(h_{ij}n^in^j)^2\rangle}{144}=\frac{\hbar}{\pi}\int_0^\infty \mathrm{Re}(\omega Z_\omega)^{-1}[1+2(\mathrm{e}^{\beta\hbar\omega}-1)^{-1}]\,\mathrm{d}\omega. \tag{3.18}$$

After averaging over all directions and polarisation states of the wave field h_{ij} and using equation (3.17) and the definition of $E(\omega,\beta)$ in § 2, we find

$$\langle h_{ij}h^{ij}\rangle=\frac{32}{\pi}\frac{G}{c^5}\int_0^\infty E(\omega,\beta)\,\mathrm{d}\omega. \tag{3.19}$$

Equation (3.19) is in complete agreement with equation (2.17).

If we insert (3.17) and $\mathrm{d}^2(3\mu x^2)/\mathrm{d}t^2\equiv Q$ into equations (3.8*a*) and (3.8*b*) we arrive, with some trivial transformations, in the limit $\beta\to\infty$ at the same quantum-mechanical quadrupole radiation damping formulae as in the papers of Schäfer and Dehnen (1980a, b). The generalised thermal formulae follow immediately from equations (3.8).

† The appearance of δ functions in the following keeps us from running into contradictions with the quadrupole approximation.

4. Discussion

Our basic result follows from the identity of formulae (2.17) and (3.19), which are valid quantum mechanically and classically (more precisely, in the classical limit; but this limit is performed here at a glance (look at $E(\omega, \beta)$)—far more immediately than in the radiation formulae (3.8)), namely, the dissipative parts in § 2 which are caused by the classical reaction force (2.10) can also be interpreted as being caused by the vacuum field fluctuations of the gravitational wave field because $\mathrm{Re}(Z_\omega)^{-1}$ has its origin in these fluctuations, as can be easily seen from equation (3.4) (take the limit $\beta \to \infty$) or equation (3.17) (state density of the vacuum gravitational field is proportional to ω^2/c^3). Contrary to this, we find a difference between the reactances following from equations (2.13) and (3.17). Whereas from equation (2.13) we get a finite reactance, that following from (3.17) is infinite. The reason is that an infinite self-energy part which is present in equation (3.17) is omitted in equation (2.13) from the beginning. All these results have their analogies in the electromagnetic dipole approximation case (e.g. equation (3.17) without the factor $G/45c^2$), which can be found in the papers by Callen and Welton (1951) and Feynman and Vernon (1963). However, what the calculations for the gravitational field, given in this paper, make especially interesting is the fact that with the identity of equations (2.17) and (3.19) we have proven a correctness for the quadrupole radiation-reaction force because the calculations which we have performed in § 3 do not suffer from the drawbacks of a matched or singular asymptotic expansion by which Misner *et al* (1973) obtained their radiation-reaction force. Beyond this, the influence functional approach of § 3 makes a clear cut between finite damping and infinite self-energy terms (real and imaginary parts of some complex quantity; see also Schäfer and Dehnen (1980b)), so that the self-energy problem must not be settled—and it is indeed not settled—to obtain reliable expressions for the damping in our approximation.

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